# Comparison of voter and Glauber ordering dynamics on networks

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We study numerically the ordering process of two very simple dynamical models for a two-state variable on several topologies with increasing levels of heterogeneity in the degree distribution. We find that the zero-temperature Glauber dynamics for the Ising model may get trapped in sets of partially ordered metastable states even for finite system size, and this becomes more probable as the size increases. Voter dynamics instead always converges to full order on finite networks, even if this does not occur via coherent growth of domains. The time needed for order to be reached diverges with the system size. In both cases the ordering process is rather insensitive to the variation of the degreee distribution from sharply peaked to scale free.

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## I. INTRODUCTION

Complex networks have become astonishingly popular in recent years as models for the interaction patterns among individuals (agents, computers, molecules, etc.) in many diverse fields [1]. The nontrivial topologies found in nature are often the structures over which dynamical processes take place. Complex interaction patterns produce in many cases nontrivial collective dynamical properties, very different from those observed when single elements are connected in a regular way.

Social systems are one of the fields where it is more evident that regular lattices are often inappropriate as models for the structure of interactions. This has led to intense activity aimed at investigating the effect of complex networks on the behavior of models for the spreading of opinions [2], the diffusion of culture [3] and other processes where domains of homogeneous individuals emerge out of an initial disordered state [4–6]. These studies have often been motivated by a direct interest in the social phenomena described by the models considered. Here we take a different point of view. We try to understand how the basic features of ordering processes occurring on complex networks depend on the topology and whether generic rules can be inferred.

For this reason we consider the simplest models that exhibit an ordering dynamics: the voter model and the Glauber-Metropolis zero-temperature dynamics for the Ising model. They are not intended to describe any real phenomenon, rather they allow the investigation of the role of simple physical ingredients (as surface tension or interfacial noise), and their interplay with the topological structure, in determining the overall behavior of the system.

For the same reason we consider extremely simple networks: the complete graph, the random graph and the scale-free Barabasi-Albert graph. We believe that our investigation provides the necessary background for a thorough understanding of more complicated and realistic dynamics on more complex networks.

As a general pattern, we find that the two types of dynamics behave rather differently on networks, while they are not much sensitive to the precise topology on which they evolve. For systems with a finite number of sites $N$, Glauber zero-temperature dynamics leads in some realizations to full ordering, while in others the system gets trapped in a set of disordered metastable configurations. The probability of not reaching order tends to grow with $N$. The voter dynamics instead always reaches the fully ordered state when the system size is finite. This however does not involve a coherent ordering process as it happens on regular lattices: the system remains on average disordered until a random fluctuation leads it to consensus. The temporal scale $\tau(N)$ needed for this process diverges as $N^{\gamma}$. The detailed form of the topology affects only the value of $\gamma$.

The paper is organized as follows. In the next section we describe the models studied and we give a brief overview of the background. Section III reports the results for voter dynamics, on the various networks considered. Glauber zero-temperature dynamics is discussed in Sec. IV. The final section contains a summary and a discussion of the results.

## II. THE MODELS

In this paper we consider two of the simplest models that, starting from a fully disordered initial state, exhibit an order-

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ing dynamics. In both cases a single variable $s_i$, that may assume only two values ($\pm 1$), fully specifies the state (opinion, culture) of site $i$.

In the zero-temperature Glauber dynamics for the Ising model, at each time step a node $i$ selected randomly and the local field is computed. Then $s_i$ is set equal to +1 if the local field is positive and to −1 if it is negative. If the field is zero the variable changes its value [7].

While the Glauber dynamics at finite temperature reaches asymptotically the state dictated by thermodynamic equilibrium, which is well known also for some complex networks [8], the situation at zero temperature is much less understood. On regular lattices it is known that, while in $d=1$ a fully ordered state is always reached, in higher dimensions the system may get stuck in a frozen state with coexisting domains of opposite magnetization. In $d=2$ this occurs with probability of about $1/3$ for large systems. In higher dimensions the probability of reaching the ground state rapidly vanishes as the system size grows and the system ends up wandering forever within an isoenergy set of metastable states [9]. The Glauber zero-temperature dynamics has been recently studied by Boyer and Miramontes [10] on the Watts-Strogatz small-world network [11]. They observe that ordering is hindered by the presence of shortcuts leading to a pinned state with a finite size of ordered domains.

The other type of dynamics we consider is the voter model: at each time step a node ($i$) and one of its neighbors ($j$) are randomly selected. Then $s_i$ is set equal to $s_j$. Also in this case the behavior on regular lattices is well known [12,13]: in $d \leqslant 2$ order is reached asymptotically and the density of interfaces decays as $t^{(d-2)/2}$; for $d>2$ a stationary active state is reached, with coexisting domains. Also this model has been studied on the small-world network of Watts and Strogatz finding that the dynamics gets stuck in a disordered stationary state on infinitely large systems, while it orders in finite systems over a time scaling as $N$ [14]. Recently, Wu and Huberman [15] and Suchecki *et al.* [16] have considered the voter dynamics on networks with heterogeneous node degrees, showing that the average magnetization is not conserved, at odds with what occurs on regular lattices. During the completion of this work a paper by Sood and Redner [17] has appeared, reporting an analytical investigation of the ordering of the voter model on heterogeneous networks. They compute the time to reach full consensus as a function of $N$ for networks with generic degree distribution.

Glauber dynamics leads to the formation of ordered domains and their coarsening as a consequence of the existence of surface tension and the drive provided by energy minimization. In the voter dynamics instead no surface tension exists and the tendency toward order is the effect of annihilation of freely diffusing interfaces between domains [18].

In the present paper the dynamical behavior of these models is studied on different types of topology. For random graph we intend an Erdos-Renyi graph: a set of $N$ nodes such that between each pair of them there is a connection with probability $p$ [19]. We have always checked that all nodes considered formed a connected cluster. For relatively small average degree $\langle k \rangle > 1$ and large system size $N$, while a giant component exists, including the overwhelming majority of nodes, isolated smaller clusters may appear. We have discarded them and run the dynamics only on the giant component of the network, always checking that at least 98% of the $N$ nodes belonged to it and that the average degree was very close to its nominal value $\langle k \rangle = p(N-1)$. To study the effect of the scale-free topology we consider the most common model known to produce such type of structure: the Barabasi-Albert graph [20].

To monitor the ordering dynamics, we focus on the density of active bonds $n_A(t)$, i.e., bonds connecting sites with opposite values of the variable $s_i$. For the Ising model it coincides with the energy density. The initial condition is always given by a completely disordered system, i.e., variables are chosen to be −1 or +1 at random in a completely uncorrelated way. Results presented are always averaged over a large number of realizations of both the topology and the initial condition.

## III. VOTER MODEL

### A. Complete graph

We start by considering a random graph in the limit $p=1$, i.e., a complete graph. In such a limit the graph is fully connected and the state of the whole system is fully specified by only one variable, the magnetization $\mu = \langle s_i \rangle$.

On such a graph, as well as on a regular lattice, the voter model coincides with the so-called "Ochrombel simplification of the Sznajd model," for which exact results have been derived by Slanina and Lavicka [21]. Writing down the master equation for the probability density $P(\mu, N_a)$ of having magnetization $\mu$ after $N_a$ attempted updates and passing to the continuum limit one obtains

$$\frac{\partial P(\mu,t')}{\partial t'} = \frac{\partial^2}{\partial \mu^2}[(1-\mu^2)P(\mu,t')], \tag{1}$$

where the natural scaling of time with the number of sites $N$, $t' = N_a/N^2$, has been introduced.

Equation (1) is a one-dimensional diffusion equation with a variable diffusion coefficient. It is solved by standard methods for the Fokker-Planck equation [21], finding that, for large $t'$, the fraction of bonds connecting nodes with opposite values of the variable (active bonds) is

$$n_A(t') = \frac{(1-m_0^2)}{2} e^{-2t'}, \tag{2}$$

where $m_0$ is the initial magnetization.

In the following we will measure time as the number of attempted updates per node $t = N_a/N$, according to the idea that each individual tries to modify its state once per unit time. In this way the voter dynamics on a complete graph has a characteristic time $\tau(N) = N/2$.

These results are exact only in the limit $N \to \infty$. As discussed in Ref. [21], for finite $N$ diffusive terms proportional to $1/N$ appear in the expansion of the master equation. Nevertheless, numerical simulations (Fig. 1) show that Eq. (2) perfectly describes the evolution of the system starting from relatively small values of $N$.

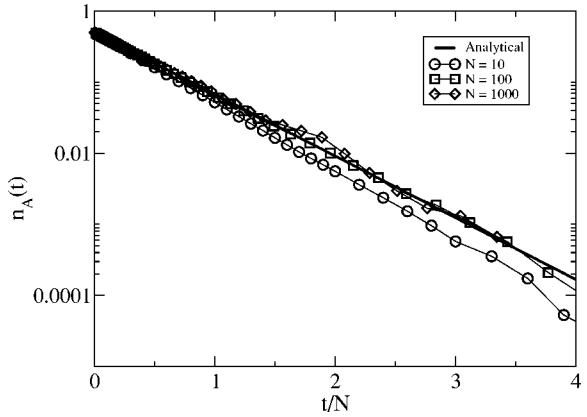


FIG. 1. Fraction of active bonds for voter dynamics on a complete graph.

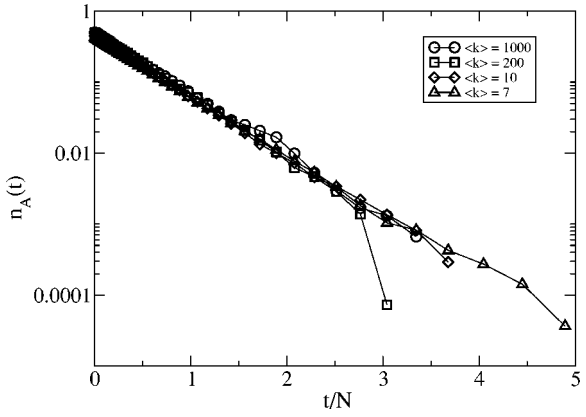


FIG. 2. Fraction of active bonds for voter dynamics on a random graph, with $N=1000$.

In order to gain further insight, let us separately consider the survival probability $\rho(t)$, i.e., the probability that a run has not reached the fully ordered state up to time $t$ and $n_A^S(t)$, the fraction of active bonds *averaged only over surviving runs*. Clearly the equality $n_A(t)=\rho(t)n_A^S(t)$ holds.

The survival probability is evaluated in Ref. [21] and it reads, for $m_0=0$,

$$\rho(t)=\begin{cases}1 & t \ll \tau(N),\\ \frac{3}{2}e^{-t/\tau(N)} & t \gg \tau(N).\end{cases} \tag{3}$$

The quantity $n_A^S(t)$ is easily computed and turns out to be

$$n_A^S(t)=\begin{cases}\frac{1}{2}e^{-t/\tau(N)} & t \ll \tau(N),\\ \frac{1}{3} & t \gg \tau(N).\end{cases} \tag{4}$$

We realize then that the fully ordered state is not reached in the thermodynamic limit. This occurs for two reasons. The first is that the temporal scale $\tau(N)$ over which consensus is reached in finite systems *diverges* with the size $N$. This happens also on regular lattices and is already evident from the behavior of $n_A(t)$. The second reason, specific to graphs, is that even for $t \gg \tau(N)$ the fraction of active bonds in surviving runs $n_A^S(t)$ does not go to zero when $N$ grows. This means that surviving runs do not order; they rather stay in configurations with, on average, a finite (and large) fraction of active bonds. Random fluctuations bring eventually all surviving runs to the fully ordered absorbing state; however, as long as the runs survive they do not order on average. The decay of $n_A(t)$ is just a consequence of the decay of $\rho(t)$, the number of such surviving runs.

This is completely different from what occurs on regular lattices. In such a case $\rho(t)$ remains 1 up to a long time, after which it quickly goes to zero. The decay of $n_A(t)$ mirrors the decay of $n_A^S(t)$: all runs survive for approximately the same time and they all get more and more ordered. An example of such fully ordering behavior is provided by Glauber dynamics on the complete graph (see below, Fig. 7).

### B. Random graph

Let us now consider what occurs for fixed $N$ and changing $p$, that is, the average degree $\langle k\rangle=p(N-1)\approx pN$ of nodes (Fig. 2). We find a remarkable similarity of the temporal evolution with the case of the complete graph, even when the average degree is changed by a factor larger than 100. Only the prefactor weakly depends on $\langle k\rangle$. The characteristic time scale $\tau(N)$ is proportional to $N$, as found in Ref. [17], and independent from the average degree $\langle k\rangle$.

Figure 3 reports the value of $n_A^S$ and $\rho(t)$ as a function of $t/N$ for several values of $N$ and $p$, with constant average degree $\langle k\rangle=10$. The behavior is very similar to what happens on a complete graph and can be summarized as follows

$$\rho(t)=\begin{cases}1 & t \ll \tau(N),\\ \frac{3}{2}e^{-t/\tau(N)} & t \gg \tau(N),\end{cases} \tag{5}$$

$$n_A^S(t)=\begin{cases}\frac{3}{2}A(\langle k\rangle)e^{-t/\tau(N)} & t \ll \tau(N),\\ A(\langle k\rangle) & t \gg \tau(N),\end{cases} \tag{6}$$

so that $\rho(t)=\frac{3}{2}A(\langle k\rangle)e^{-t/\tau(N)}$ for all times. The prefactor $A(\langle k\rangle)$ is equal to $1/3$ for the fully connected graph, while it is smaller for finite $\langle k\rangle$. We can conclude that, on random as on complete graphs, surviving runs do not get ordered.

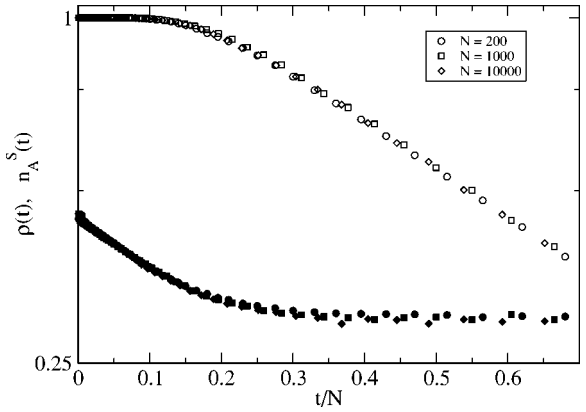


FIG. 3. Fraction of active bonds in surviving runs $n_A^S(t)$ (filled symbols) and survival probability $\rho(t)$ (empty symbols) for voter dynamics on a random graph for $\langle k\rangle=10$.

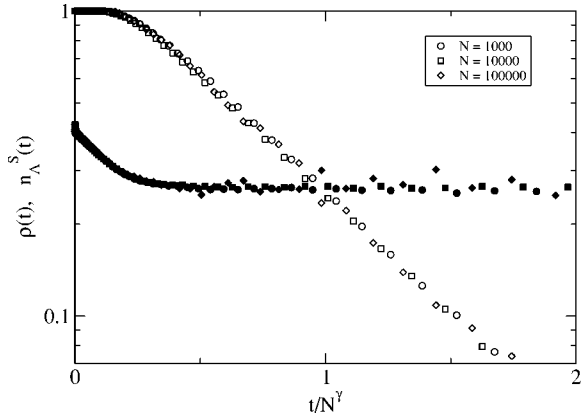


FIG. 4. Fraction of active bonds in surviving runs $n_A^S(t)$ (filled symbols) and survival probability $\rho(t)$ (empty symbols) for voter dynamics on a Barabasi-Albert graph for $m=3$.

### C. Barabasi-Albert graph

We consider then the voter model evolving on a scale-free graph built according to the rules of Barabasi and Albert (BA) [20]. These graphs are constructed by considering an initial fully connected core of $m+1$ nodes and iteratively adding new nodes, each with $m$ bonds. The other node to which a new bond is linked is chosen among existing nodes with probability proportional to their degree (preferential attachment).

Also on the BA networks the fraction of active bonds $n_A(t)$ as a function of time decays, for $m>1$, exponentially fast to zero. The survival probability $\rho(t)$ and the fraction of active bonds restricted to surviving runs $n_A^S(t)$ (Fig. 4) follow Eqs. (3) and (4), i.e., on finite systems the model always reaches the perfectly ordered state, but surviving runs do not order. The only difference is the scaling of time with the number $N$ of nodes, which is reported in Fig. 5. A power-law fit yields, independently from $m>1$, $\tau(N)\sim N^{\gamma}$, with $\gamma=0.880\pm0.003$. The nontrivial scaling of $\tau(N)$ with $N$ had already been observed by Suchecki *et al.* [16], which estimated $\gamma$ by fitting $\tau(N)$ over a decade. Here we find a compatible value over more than 3 decades. In Ref. [17], Sood and Redner estimate analytically $\tau(N)=N/\log(N)$ for this case. In Fig. 5, we compare this expression to numerical data, finding a good agreement, but no sign of the increase of the effective exponent, which would be the signature of the logarithmic correction.

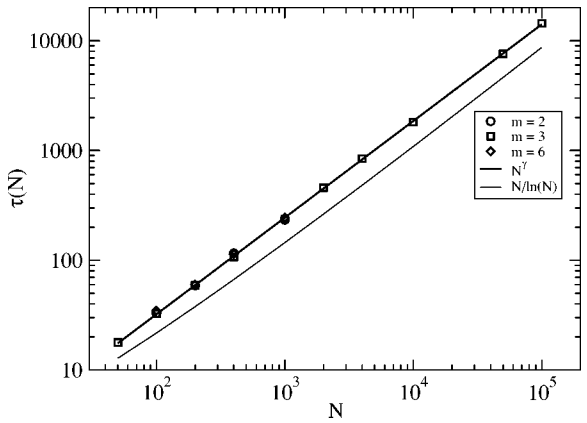


FIG. 5. Scaling of the time $\tau(N)$ vs $N$ on Barabasi-Albert graphs for several values of $m$, with a pure power-law fit with $\gamma=0.880$ (thick line) and the formula $N/\log(N)$ (thin line).

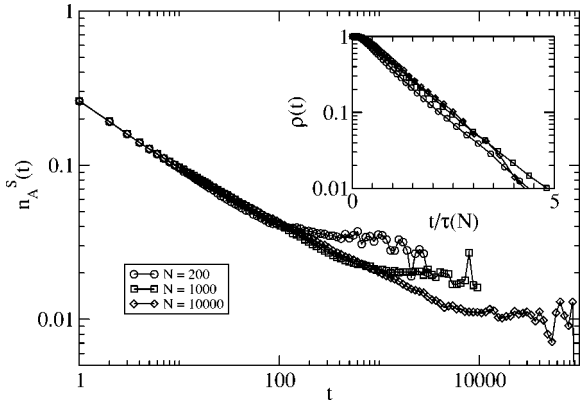


FIG. 6. Fraction of active bonds in surviving runs $n_A^S(t)$ (main) and survival probability $\rho(t)$ (inset) for voter dynamics on a Barabasi-Albert graph with $m=1$.

Also for $m=1$ consensus is reached on finite systems, but much more slowly: the decay of $n_A^S(t)$ is not exponential; it is power law or, possibly, even slower since the exponent is close to $1/3$ on the accessible temporal scales but it seems to be decreasing with time (Fig. 6). Furthermore, the plateau of $n_A^S(t)$ weakly depends on $N$. This is probably a preasymptotic effect. The survival probability $\rho(t)$ (Fig. 6, inset) decays exponentially over a temporal scale $\tau(N)\sim N^{\gamma}$ with $\gamma=1.04\pm0.01$. The differences with respect to the case $m>1$ are a consequence of the treelike structure of the BA network for $m=1$.

In order to investigate the universality of the exponent $\gamma$ we finally study the ordering dynamics of the voter model on a network built according to the prescriptions of Ref. [22]. This graph is grown by iteratively adding nodes. Each of them is connected to the nodes linked by $m$ randomly chosen edges. In this way a preferential attachment mechanism is implemented so that this network has topological properties practically identical to the one by Barabasi-Albert, with the notable exception of a large clustering coefficient [23]. This variation has little impact on the ordering dynamics. For $m>1$, the phenomenology is exactly the same as the BA graph, with the sole difference that $\gamma=0.978\pm0.005$.

In summary we find that the voter dynamics on the scale-free networks with $m>1$ shows a remarkable similarity with the dynamics on random and complete graphs. The nontrivial topology of the BA graph is reflected only in the scaling of the characteristic time with the number of sites, which follows a different power law.

## IV. GLAUBER $T=0$ DYNAMICS

### A. Complete graph

On the complete graph, also Glauber dynamics can be solved analytically in the limit $N\to\infty$. The master equation

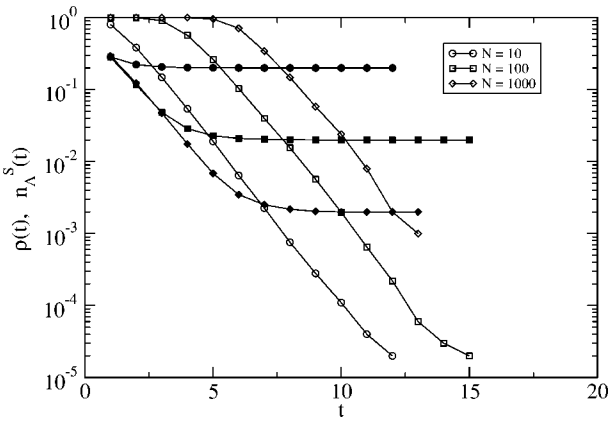


FIG. 7. Fraction of active bonds in surviving runs $n_A^S(t)$ (filled symbols) and survival probability $\rho(t)$ (empty symbols) for Glauber $t=0$ dynamics on a complete graph.

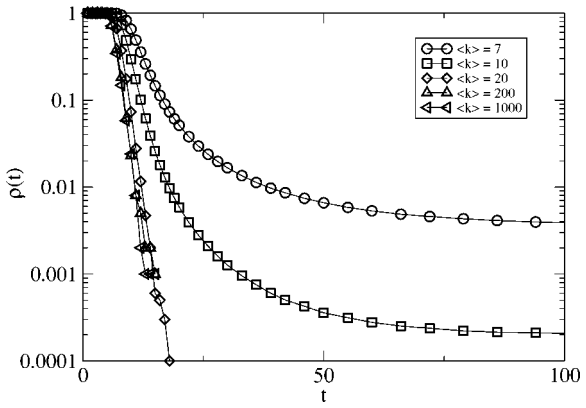


FIG. 8. Survival probability for Glauber $T=0$ dynamics on a random graph with $N=1000$ for different values of the average degree $\langle k\rangle$ of nodes.

for the probability $P(q,t)$ of having a fraction $q$ of positive spins at time $t$, is, in the continuum limit, for $q>1/2$,

$$\frac{\partial P(q,t)}{\partial t} = -\frac{\partial}{\partial q}[(1-q)P(q,t)], \tag{7}$$

where the natural definition of time is $t=N_a/N$.

The ansatz $P(q,t)=F(q,t)/(1-q)$ leads to the expression

$$P(q,t) = \frac{1}{(1-q)}F\left[\frac{e^{-t}}{(1-q)}\right]. \tag{8}$$

Hence the form of $P$ remains, during the temporal evolution, equal to the initial condition. If the initial condition is a $\delta$-function in $q=q_0$, then $P(q,t)=\delta(q-\langle q\rangle)$ where $\langle q\rangle = 1-(1-q_0)e^{-t}$.

The fraction of active bonds is then

$$n_A(t) = 2\langle q\rangle(1-\langle q\rangle) = 2\left[1-(1-q_0)e^{-t}\right](1-q_0)e^{-t}. \tag{9}$$

The comparison with numerical simulations shows perfect agreement already for $N$ of the order of 50. The exponential decay of $n_A(t)$ is perfectly similar to what happens for the voter model on a complete graph. But if we consider separately the fraction of active bonds in surviving runs $n_A^S(t)$ and the survival probability $\rho(t)$ (Fig. 7) we find a picture quite different from the case of voter dynamics [Eqs. (3) and (4)]. The fraction of active bonds for surviving run decays exponentially and then reaches a plateau, but the height of the plateau depends on $N$ and goes to zero as $N\to\infty$. This is analogous to what occurs on regular lattices and it means that the Glauber dynamics is effective at ordering the Ising model on a complete graph.

### B. Random graph

Let us now consider what occurs for fixed $N$ and changing $p$. The first change is exhibited by the survival probability in Fig. 8. While for large $\langle k\rangle$ the decay is exponential, for smaller values of $\langle k\rangle$ a plateau appears, indicating that not all realizations of the dynamics end up in an ordered state, i.e., with all nodes sharing the same value of the variable $s_i$. In such runs the system remains trapped forever in configurations with part of the nodes with $s_i=-1$ and the rest with $s_i=1$.

A freezing in a disordered state for Glauber dynamics on a random graph had already been noticed by Svenson [24] and considered analytically by Häggström [25], who showed that, in the limit $N\to\infty$, the dynamics fails to reach the global energy minimum (ordered state) for any $\langle k\rangle>0$. What is the origin of this behavior?

This phenomenon is not related to special realizations of the random graph topology. If we fix the topology and let the dynamics evolve many times on it, we see that in the same finite fraction of runs the system reaches a disordered state, independently from the particular realization of the topology.

One could think that, given the low value of $p$, there may be small "communities" in the graph, i.e., groups of nodes, tightly bound with each other with only few connections with the rest of the system. Such communities could become ordered and be basically decoupled from the rest of the system, leading to a frozen disordered state essentially made by a huge ordered set and few small chunks ordered in the opposite way. While this may be true for small $\langle k\rangle$ and large $N$, here the explanation is different. The total magnetization $[(1/N)\Sigma_i s_i]$ in the disordered state is always very close to zero and the number of domains present is always equal to 2. Moreover the large $t$ limit of the fraction of active bonds in surviving runs $n_A^S(t)$ indicates that a very high fraction of the bonds connects sites with different values of $s_i$ (Fig. 9). Hence we can conclude that the system remains trapped in configurations with two highly intertwined domains of roughly the same size. The asymptotic value of $n_A(t)$ is much higher than the lower bound computed in Ref. [25], $\langle k\rangle e^{-6\langle k\rangle}/256$.

In order to characterize further the dynamics we report in Fig. 10 the average degree of nodes that flip at time $t$ for some values of $p$ and $N$. It turns out that the disordered state is *not frozen*. It is instead a stationary active state, with some spins flipping, while keeping the energy conserved. The qualitative picture is then the same holding on regular lattices for $d>2$: the system wanders forever in an isoenergy set of states.

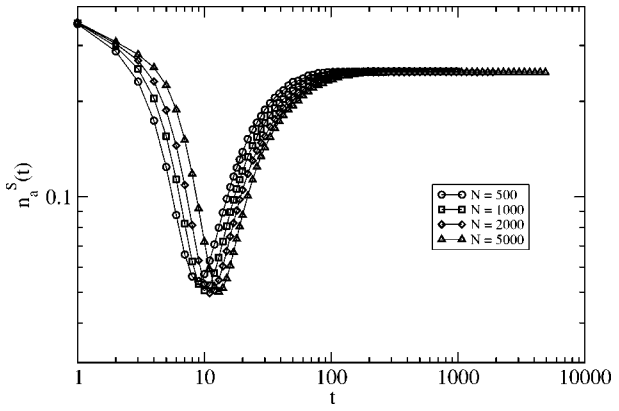


FIG. 9. Value of the fraction of active bonds in surviving runs $n_A^S(t)$ as a function of $t$ for several values of $N$ and $\langle k\rangle=7$.

It is finally interesting to consider the probability $p_{\rm dis}$ that this stationary disordered state occurs, i.e., the limit of $\rho(t)$ for $t\to\infty$. As shown in Fig. 11, this probability has a nontrivial behavior as a function of $N$ for fixed $\langle k\rangle$: for very small and large $N$ it grows, while it has an intermediate regime such that it decreases as the system is made bigger. Although Ref. [25] guarantees that $p_{\rm dis}$ goes to 1 as $N$ diverges, for reasonable values of $N$ the values of $p_{\rm dis}$ are much smaller than one.

We have no clear understanding of the reason for the nonmonotonic behavior. It is probably related in some way to the connectivity transition that occurs in random graphs for $p=p_1=\ln(N)/N$ [19]. For $p>p_1$ all nodes belong (in the limit $N\to\infty$) to the giant component, while for $p<p_1$ separate components exist. If we invert this relation we obtain an expression for the value of $N$ where the connectivity transition occurs in terms of $\langle k\rangle$: $N_m=e^{\langle k\rangle}$. Then for $N<N_m(\langle k\rangle)$ only the giant component exists, while for $N>N_m(\langle k\rangle)$ some nodes belong to disconnected clusters. It is tempting to associate $N_m(\langle k\rangle)$ with the value of $N$ such that $p_{\rm dis}$ is minimal. The data presented in Fig. 11 are compatible with this picture, though we do not have a valid explanation why the decrease of $p_{\rm dis}$ would correspond to the case where only the giant component exists.

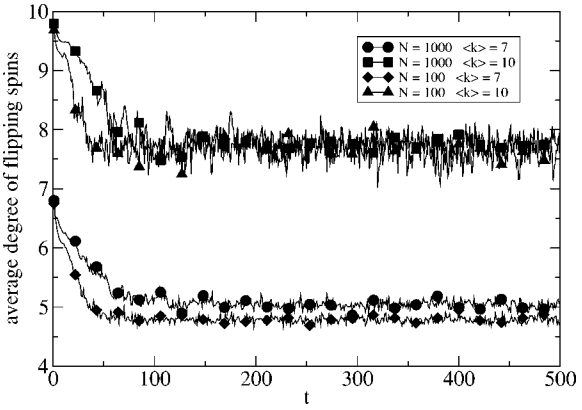


FIG. 10. Average degree of spins that flip at time $t$ for several values of $\langle k\rangle$ and $N$.

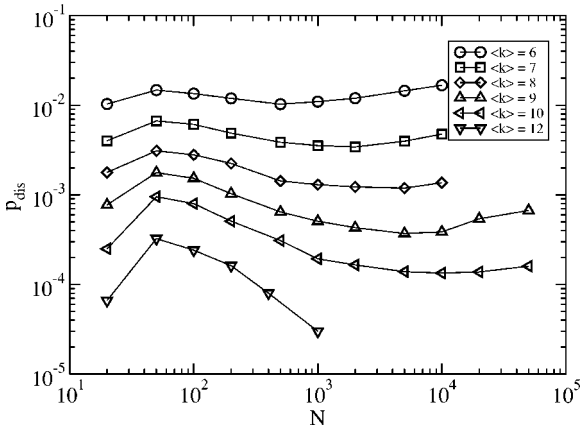


FIG. 11. Probability of ending in the disordered state as a function of $N$ for several values of $\langle k\rangle$.

### C. Barabasi-Albert graph

On Barabasi-Albert networks, the global behavior of Glauber $T=0$ dynamics is similar to the one exhibited on random graphs: in a fraction of the runs the system reaches a disordered stationary state with two domains of opposite magnetization: spins continue to flip but the energy does not decrease further.

In Fig. 12 we report the fraction of such runs as a function of the number of nodes $N$, for several values of the average degree $\langle k\rangle=2m$, where $m$ is the number of edges added for each new node. In analogy with the case of the random graph, the probability of remaining disordered grows for large values of $N$ except for the case $\langle k\rangle=12$, where a decrease is seen. This is similar to what happens on a random graph for large average degree. For $\langle k\rangle=2$ the probability rapidly reaches the value 1, i.e., no run reaches full order. This can be easily understood given the treelike structure of the Barabasi-Albert graph for $m=1$.

## V. CONCLUSIONS

In summary, we have investigated the behavior of the simplest ordering dynamics for a two-valued variable on networks ranging from the fully connected graph to random and

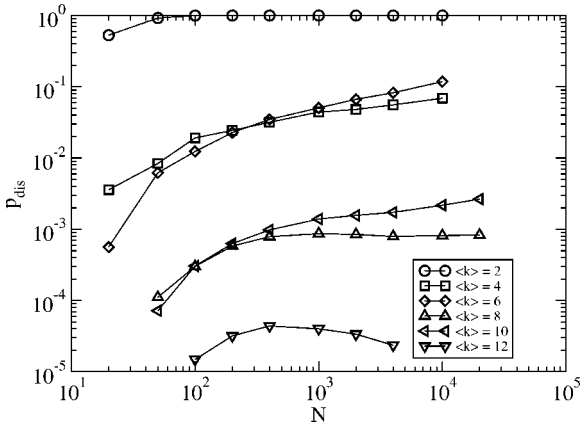


FIG. 12. Probability that dynamics reaches a disordered state on the Barabasi-Albert graph as a function of $N$ for the zero-temperature Glauber dynamics.

scale-free graphs. In general we find that the difference between the zero-temperature Glauber-Metropolis Ising dynamics and the voter model has a quite strong impact on the ordering process starting from a completely random initial condition. On the other hand, the presence or absence of a typical scale in the network describing the interaction pattern has a limited effect: it changes (in the voter model) the temporal scale over which order is reached, but does not affect whether or not such order is reached.

The voter model dynamics invariably leads to full ordering, for any type of topology, provided the number $N$ of nodes in the system is finite. When $N$ grows the time needed to reach complete consensus diverges, in a way that depends on the connectivity pattern. If one considers the number of active bonds $n_A(t)$ as a function of time averaged over all realizations of the dynamics, including those that already have reached the fully ordered state, one sees an exponential decay. This may lead to the conclusion that the system actually orders exponentially fast (i.e., faster than on regular lattices). However, this conclusion disagrees with the connection between the voter dynamics and the recurrence properties of random walks. The recurrence of the random walk on regular lattices for $d \leqslant 2$ implies that the voter model orders on them, while it remains in a disordered state when the walk is transient ($d > 2$). The same argument implies that the voter model does not order on networks, as those considered here, for which the random walk is transient [26]. The solution of this apparent paradox is that the voter model actually does not get ordered on networks in the thermodynamical limit. The right quantity to observe this is the density of active bonds in surviving runs $n_A^S(t)$, which does not decay to zero; it attains large values, signaling that in surviving runs the system is again split in two domains with a large number of interconnections. Notice that this is true also for the complete graph.

For Glauber dynamics on a complete graph full ordering is attained for any system size, including the thermodynamic limit. Randomness in the connectivity pattern implies instead that even a finite system has a nonzero probability to remain trapped in metastable states, i.e., to indefinitely cycle through configurations with the same energy. In such metastable configurations, the system is split in two domains with a large number of interconnections. When the system size grows the probability of reaching this disordered stationary state tends to increase, making full ordering less likely. Apart from details, this holds true for both random (Erdös-Renyi) and scale-free (Barabasi-Albert) graphs.

We are now in the position to summarize the common features and the differences between the behavior of the voter model and the Glauber zero-temperature dynamics. At the level of the complete graph, despite the apparent similarity between Eq. (2) and Eq. (9), the two models are different, since Glauber reaches genuine order in the thermodynamic limit, while voter does not. In the presence of a random topology, the similarity is stronger: In both cases, the system initially approaches a state with two intertwined domains of roughly the same size. For the voter model, the large noise present in the dynamics creates fluctuations that eventually lead, in finite systems, to complete ordering. In the Glauber case, instead, the zero-temperature condition forbids energy fluctuations and the dynamics remains confined to isoenergetic partially ordered configurations.

The two dynamical models considered are not dramatically sensitive to the underlying topology. The existence or absence of a characteristic scale in the degree distribution of the network does not affect whether order is reached or not. For the voter dynamics, the degree distribution only affects the way the characteristic temporal scale $\tau$ depends on $N$. We have shown that for over three decades the exponent $\gamma = 0.880 \pm 003$ fits very well the numerical data, but this does not rule out the analytical prediction $\tau \sim N/\log N$ of Ref. [17]. However, the results for the scale-free graph by Dorogovtsev, Mendes and Samukhin seem to indicate that the precise value of $\gamma$ may depend on other details of the underlying network.

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